NETWORK THEOREMS

1. Thevenin’s theorem

Statement: A linear network consisting of a number of voltage sources and resistances can be replaced by an equivalent network having a single voltage source called Thevenin’s voltage \( V_{th} \) and a single resistance called Thevenin’s resistance \( R_{th} \).

Explanation:

Consider a network or a circuit as shown. Let \( E \) be the emf of the cell having its internal resistance \( r = 0 \). \( R_L \rightarrow \) load resistance across \( AB \).

To find \( V_{th} \):

\[
V_{th} = I R_2 \quad \Rightarrow \quad V_{th} = \frac{E R_2}{R_1 + R_2}
\]

To find \( R_{th} \):

The load resistance \( R_L \) is removed. The cell is disconnected and the wires are short as shown.

The effective resistance across \( AB = \) Thevenin’s resistance \( R_{th} \).

\[
R_{th} = R_3 + \frac{R_1 R_2}{R_1 + R_2} \quad [R_1 \text{ is parallel to } R_2 \text{ and this combination in series with } R_3]
\]

If the cell has internal resistance \( r \), then \( V_{th} = \frac{E R_2}{R_1 + R_2 + r} \) and \( R_{th} = R_3 + \frac{(R_1 + r) R_2}{R_1 + r + R_2} \).
Proof of Thevenin’s theorem:
Consider the network as shown below

The equivalent circuit is given by

The effective resistance of the network in (1) is \( R_3 \) and \( R_L \) in series and this combination is parallel to \( R_2 \) which in turn is in series with \( R_1 \).

Thus, \( R_{\text{eff}} = R_1 + \frac{R_2(R_3 + R_L)}{R_2 + R_3 + R_L} \) \hspace{1cm} (1)

The current \( I \) in the circuit is

\[
I = \frac{E}{R_{\text{eff}}} = \frac{E}{R_1 + \frac{R_2(R_3 + R_L)}{R_2 + R_3 + R_L}}
\]

or

\[
I = \frac{E(R_2 + R_3 + R_L)}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L} \quad (2)
\]

The current through the load resistance \( I' \) is found using branch current method.

\[
I' = \frac{IR_2}{R_2 + R_3 + R_L} \quad (3)
\]

Substituting for \( I \) from (2) in (3)

\[
I' = \frac{E(R_2 + R_3 + R_L)R_2}{(R_2 + R_3 + R_L)(R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L)}
\]

or

\[
I' = \frac{ER_2}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L} \quad (4)
\]

Thevenin’s voltage \( V_{\text{th}} = \frac{ER_2}{R_1 + R_2} \) \hspace{1cm} (5)

Thevenin’s resistance \( R_{\text{th}} = R_3 + \frac{R_1R_2}{R_1 + R_2} \) \hspace{1cm} (6)

Consider the equivalent circuit (circuit (2))

The current \( I'' \) in the equivalent circuit is

\[
I'' = \frac{V_{\text{th}}}{R_{\text{th}} + R_L} \quad (7)
\]
Substituting for \( V_{th} \) and \( R_{th} \) from (5) and (6) in (7)

\[
I'' = \frac{\frac{1}{R_3} + \frac{R_1 R_2}{R_1 + R_2} + R_L}{R_3} \left( \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_1 R_2 + R_1 R_L + R_2 R_3 + R_1 R_3 + R_2 R_L + R_3 R_L \right) = \frac{E R_2}{R_2 R_3 + R_3 R_1 + R_1 R_2 + R_2 R_L + R_1 R_L + R_3 R_L} \quad \text{(8)}
\]

or

\[
I'' = \frac{E R_2}{R_2 R_3 + R_3 R_1 + R_1 R_2 + R_2 R_L + R_1 R_L + R_3 R_L}
\]

From equations (4) and (8), it is observed that \( I' = I'' \).

Hence Thevenin’s theorem is verified.

2. Maximum Power Transfer Theorem

**Statement:** The power transferred by a source to the load resistance in a network is maximum when the load resistance is equal to the internal resistance of the source.

**Proof of Maximum power transfer theorem:**

Consider a network with a source of emf \( E \) and internal resistance \( r \) connected to a load resistance \( R_L \). The current \( I \) in the circuit is

\[
I = \frac{E}{R_L + r} \quad \text{(1)}
\]

The power delivered to load resistance \( R_L \) is \( P_L = I^2 R_L \) or \( P_L = \left( \frac{E}{R_L + r} \right)^2 R_L \)

(from equation (1))

\[
P_L = \frac{E^2}{(R_L + r)^2} R_L \quad \text{(2)}
\]

The variation of \( P_L \) with \( R_L \) is as shown.

\( P_L \) is found to be maximum for a particular value of \( R_L \) when

\[
\frac{dP_L}{dR_L} = 0
\]

[\( \because \) No variation of \( P_L \) with \( R_L \) at \( P_{L_{\text{max}}} \)]

i.e.,

\[
\frac{d}{dR_L} \left( \frac{E^2 R_L}{(R_L + r)^2} \right) = 0 \quad \text{or} \quad \frac{E^2 R_L}{(R_L + r)^2} \left[ \frac{2R_L}{(R_L + r)^2} \right] = 0
\]

Differentiating

\[
\frac{2R_L}{(R_L + r)^2} \left[ R_L (-2)(R_L + r)^3 + (R_L + r)^2 \right] = 0 \quad \text{or} \quad \frac{-2R_L}{(R_L + r)^3} + \frac{1}{(R_L + r)^2} = 0
\]

or

\[
\frac{2R_L}{(R_L + r)^2} = \frac{1}{(R_L + r)^2}
\]

Thus,

\[
\frac{2R_L}{R_L + r} = 1 \quad \Rightarrow \quad 2R_L = R_L + r \quad \text{or} \quad R_L = r
\]

Thus the power delivered to the load resistance is maximum when the load resistance is equal to the internal resistance of the source.
To show that the maximum power transfer efficiency of a circuit is 50%:

The power across the load \( P_L = I^2 R_L = \frac{E^2}{(R_L + r)^2} R_L \) \( \text{-------- (1)} \)

From the maximum power transfer theorem, \( P_L \) is maximum when \( R_L = r \). Putting this condition in equation (1),

\[ P_{L_{\text{max}}} = \frac{E^2}{4R_L} \text{ max} \]

The power that is taken from the voltage source is (or power generated by the source),

\[ P = \frac{E^2}{2R_L} \text{ or } P = \frac{E^2}{R_L + r} \text{ When } R_L = r, \]

Dividing equation (2) by (3)

\[ \frac{P_{L_{\text{max}}}}{P} = \frac{\frac{E^2}{4R_L} \times \frac{2R_L}{E^2}}{\frac{E^2}{2R_L}} = \frac{1}{2} \text{ or } \frac{P_{L_{\text{max}}}}{P} = \frac{1}{2} \]

Thus the maximum power delivered to the load is only half the power generated by the source or the maximum power transfer efficiency is 50%. The remaining 50% power is lost across the internal resistance of the source.

3. Superposition theorem

**Statement:** In a linear network having number of voltage or current sources and resistances, the current through any branch of the network is the algebraic sum of the currents due to each of the sources when acting independently.

**Explanation:** By mesh current analysis.

1. Consider the network as shown. The currents in different branches of the network are \( I_1, I_2 \) and \( I \) as shown. Also \( I_1 + I_2 = I \).

2. [Let the internal resistance \( r \) of the cells be negligible].

   The cell \( E_2 \) is removed and the terminals are short as shown. Now the currents in the branches are \( I_1', I_2' \) and \( I' \). Also \( I' = I_1' + I_2' \).
3. The $E_1$ is removed and the terminals are short as shown. The currents are $I_1'', I_2''$ and $I''$. Also $I'' = I_1'' + I_2''$. 

![Network Theorems Diagram]

According to superposition theorem $I_1 = I_1' + I_1''$. $I_2 = I_2' + I_2''$ and $I = I' + I''$

$I = I_1 + I_2$

**Verification of superposition theorem:**

1. Consider the network shown. Applying Kirchoff’s voltage to the loop 1.

$I_1R_1 + I_3R_3 + I_2R_3 = E_1$ 

or $I_1 = \frac{E_1 - I_3R_3}{R_1 + R_3}$

Considering loop 2, $I_1R_1 + I_3R_3 = E_2$

$I_2 = \frac{E_2 - I_1R_3}{R_2 + R_3}$

Thus, $I = I_1 + I_2$

$I = \frac{E_1 - I_3R_3}{R_1 + R_3} + \frac{E_2 - I_1R_3}{R_2 + R_3}$

2. Consider the circuit shown with $E_2$ removed and terminals short. Applying Kirchoff’s law to loop 1.

$I_1R_1 + I'R_3 = E_1$ 

As $I' = I_1' + I_2'$,

$I_1R_1 + I_1'R_3 + I_2'R_3 = E_1 \Rightarrow I_1' = \frac{E_1 - I_2'R_3}{R_1 + R_3}$

Similarly for loop 2,

$I_2' = \frac{-I_1'R_3}{R_2 + R_3}$

$I' = I_1' + I_2' = \frac{E_1 - I_2'R_3}{R_1 + R_3} - \frac{I_1'R_3}{R_2 + R_3}$

3. Consider the circuit with $E_1$ removed and terminals short.

For loop (1) $I_1''R_1 + I''R_3 = 0$
As \( I'' = I_1'' + I_2'' \)

\[ I_1''R_1 + I_1''R_3 + I_2''R_3 = 0 \quad \Rightarrow \quad I_1'' = -\frac{I_2''R_3}{R_1 + R_3} \quad \text{(7)} \]

For loop (2)

\[ I_2''R_2 + I''R_3 = E_2 \quad \Rightarrow \quad I_2''R_2 + I_1''R_3 + I_2''R_3 = E_2 \]

or \( I_2'' = \frac{E_2 - I_1''R_3}{R_2 + R_3} \quad \text{(8)} \)

\[ I'' = I_1'' + I_2'' = -\frac{I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3} \quad \text{(9)} \]

Adding equations (6) and (9)

\[ I'' = \frac{E_1 - I_2'R_3}{R_1 + R_3} - \frac{I_1'R_3}{R_2 + R_3} - \frac{I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3} \]

\[ = \frac{1}{R_1 + R_3} \left[ E_1 - I_2'R_3 - I_1'R_3 \right] + \frac{1}{R_2 + R_3} \left[ E_2 - I_1''R_3 - I_1'R_3 \right] \]

\[ I' + I'' = \frac{1}{R_1 + R_3} \left[ E_1 - R_3 (I_2' + I_2'') \right] + \frac{1}{R_2 + R_3} \left[ E_2 - R_3 (I_1' + I_1'') \right] \quad \text{(10)} \]

Comparing equations (3) and (10) it is observed that

\[ I_1 = I_1' + I_1'' \]

\[ I_2 = I_2' + I_2'' \]

\[ I = I' + I'' \]

Hence the proof of the theorem.